Contemporary Mathematics

Level of Siegel modular forms constructed via sym^3 lifting

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Abstract. Ramakrishnan-Shahidi proved a lifting from a non-CM elliptic curve E over $\mathbb{Q}$ to a degree 2 Siegel cusp form F of weight 3. We want to better understand the level of the Siegel cusp form F coming from their lifting. Moreover, we are interested in the level of F with respect to different congruence subgroups.

1. Introduction

Kim and Shahidi [**KS02**] proved the Langlands functoriality from $\mathrm{GL}(2,\mathbb{A})$ to $\mathrm{GL}(4,\mathbb{A})$ coming from the symmetric cube (sym^3) map from $\mathrm{GL}(2,\mathbb{C})$ to $\mathrm{GL}(4,\mathbb{C})$. Ramakrishnan and Shahidi [**RS07a**] proved the following theorem using the sym^3 lifting, which generalizes the lifting from a non-CM elliptic curve to a Siegel cusp form of degree 2 and weight 3.

Theorem 1.1. *Let $\pi \cong \bigotimes_p \pi_p$ be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ defined by a holomorphic, non-CM newform f of even weight $k \geq 2$, level N with trivial central character. Then there exists a cuspidal automorphic representation $\Pi \cong \bigotimes_p \Pi_p$ of $\mathrm{GSp}(4,\mathbb{A})$ with trivial central character, which is unramified at any prime p not dividing N, such that*

1. *Π_∞ is a holomorphic discrete series representation, with its parameter being sym^3 of the archimedean parameter of π.*
2. *$\mathrm{L}(\mathrm{s},\Pi) = \mathrm{L}(\mathrm{s},\pi,\mathrm{sym}^3)$.*

In the above theorem, Ramakrishnan and Shahidi lift a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ to a cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$. First, they use the functoriality from $\mathrm{GL}(2,\mathbb{A})$ to $\mathrm{GL}(4,\mathbb{A})$. Then via the descent method they obtain a cuspidal automorphic representation of $\mathrm{SO}(5,\mathbb{A})$ from $\mathrm{GL}(4,\mathbb{A})$. Then they construct a cuspidal automorphic representation of $\mathrm{Sp}(4,\mathbb{A})$ from the cuspidal automorphic representation of $\mathrm{SO}(5,\mathbb{A})$ and finally they get a cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ from the representation of $\mathrm{Sp}(4,\mathbb{A})$.

By this construction they get a globally generic representation of $\mathrm{GSp}(4,\mathbb{A})$. To switch the archimedean component Π_∞ from generic to a holomorphic discrete series, they give a proof using ℓ-adic cohomology.

1991 *Mathematics Subject Classification.* Primary 11F46, 11F70;

2. Current Work

2.1. Different proof of the theorem. Using results that were not available at the time, we can now give a streamlined proof of the Ramakrishnan-Shahidi theorem involving the following groups, dual groups and liftings.

$$\begin{array}{ccc} \mathrm{GL}(2,\mathbb{A}) & \xrightarrow{\text{functoriality}} & \mathrm{GSp}(4,\mathbb{A}) \\ \downarrow & & \uparrow \\ \mathrm{GL}(4,\mathbb{A}) & \underset{\text{functoriality}}{\overset{\text{descent}}{\rightleftarrows}} & \mathrm{SO}(5,\mathbb{A}) \end{array} \qquad \begin{array}{ccc} \mathrm{GL}(2,\mathbb{C}) & \xrightarrow{\mathrm{sym}^3} & \mathrm{GSp}(4,\mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{sym}^3} & & \uparrow \\ \mathrm{GL}(4,\mathbb{C}) & \longleftarrow & \mathrm{Sp}(4,\mathbb{C}) \end{array}$$

The image under the sym^3 map lies in $\mathrm{GSp}(4,\mathbb{C})$. We can see that the lifting from $\mathrm{GL}(2,\mathbb{A})$ to $\mathrm{GSp}(4,\mathbb{A})$ via sym^3 is functorial at each place, since all other liftings in the above diagram respect functoriality at each place and the local Langlands correspondence is true for $\mathrm{GL}(2)$ and $\mathrm{GSp}(4)$. We use Arthur's packet structure to see that the representation of $\mathrm{GSp}(4,\mathbb{A})$ is of "general" type, which helps us to prove the claims of the theorem.

2.2. Level under the paramodular group. The cuspidal automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$ are connected to the theory of Siegel cusp forms. One can consider the Siegel cusp forms coming from the sym^3 lifting. We are interested in finding the level of these Siegel cusp forms under some suitable congruence subgroups. In the paper [**RS07a**], the level is measured in terms of principal congruence subgroups. However, for a correspondence between cuspidal automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$ and Siegel cusp forms, it is more convenient to consider a different congruence subgroup known as the paramodular group:

$$K(N) = \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

There is a well understood connection between paramodular forms (Siegel modular forms with respect to the paramodular group) and cuspidal automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$, and there is a nice newform theory for paramodular forms [**RS07b**]. We now focus on finding the level of the paramodular forms obtained by the sym^3 lifting. The following result, one of the results in [**Roy**], is about the paramodular forms coming from elliptic curves via the sym^3 lifting:

THEOREM 2.1. *Given a non-CM elliptic curve E over $\mathbb{Q}$ with the global minimal Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ with coefficients a_1, a_2, a_3, a_4, a_6 in $\mathbb{Z}$, there exists a cuspidal paramodular newform F of weight 3 and level M such that*

$$L(s,F) = L(s,E,\mathrm{sym}^3),$$

and the level M can be determined in an explicit and elementary way from the given Weierstrass coefficients a_1, a_2, a_3, a_4, a_6 of E.

In the following table, we give some examples to illustrate Theorem 2.1.

$E/\mathbb{Q}$	Conductor of E	Level of F
$y^2 + y = x^3 - x^2$	11	$11^3 = 1331$
$y^2 + xy + y = x^3 - x - 2$	$50 = 2 \cdot 5^2$	$2^3 \cdot 5^2 = 200$
$y^2 + xy + y = x^3 + x^2 - 3x + 1$	$50 = 2 \cdot 5^2$	$2^3 \cdot 5^4 = 5000$
$y^2 + xy = x^3 - x^2 - 3x + 3$	$54 = 2 \cdot 3^3$	$2^3 \cdot 3^5 = 1944$
$y^2 + xy = x^3 + x^2 - 2x - 7$	$121 = 11^2$	$11^2 = 121$
$y^2 + y = x^3 - x^2 - 7x + 10$	$121 = 11^2$	$11^4 = 14641$
$y^2 + xy + y = x^3 - x^2 - 5x + 5$	$162 = 2 \cdot 3^4$	$2^3 \cdot 3^4 = 648$
$y^2 + xy = x^3 - x^2 + 3x - 1$	$162 = 2 \cdot 3^4$	$2^3 \cdot 3^6 = 5832$
$y^2 + xy = x^3 - x^2 + 3x + 5$	$486 = 2 \cdot 3^5$	$2^3 \cdot 3^7 = 17496$

Note that we may have different paramodular levels for elliptic curves with the same conductor. This indicates that the level of the paramodular form F does not only depend on the conductor E, but on some data involving the coefficients of the Weierstrass equation of E.

References

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